Chapter 2

LONGITUDINAL PHASE SPACE

2.1 Momentum Compaction

A bunch of charged particles has a spread of energy because of many reasons, for example, random quantum excitation which changes the energy of the particles randomly (for electrons and ultra-high energy protons only), intrabeam scattering which is just Coulomb scattering among the particles, Touschek scattering [1] which is large-angle Coulomb scattering which converts the transverse momentum of a particle into longitudinal, and, most important of all, a means to counter collective instabilities through Landau damping. In an accelerator ring or storage ring, particles with different energies have different closed orbits, their lengths are given by

$$C = C_0 \left[1 + \alpha_0 \delta + \mathcal{O}(\delta^2) \right] \ , \tag{2.1}$$

where δ is the fractional spread in momentum and C_0 is the orbit length of the so-called *on-momentum* particle. The proportionality constant α_0 is called the *momentum-compaction factor* of the accelerator ring. The fraction momentum spread is related to the lowest order fractional energy spread $\Delta E/E_0$ by

$$\delta = \frac{\Delta p}{p_0} \approx \frac{1}{\beta_0^2} \frac{\Delta E}{E_0} \ . \tag{2.2}$$

where p_0, E_0, and $v_0 = \beta_0 c$ are the momentum, energy, and longitudinal velocity of the on-momentum particle. The momentum-compaction factors of most accelerators and storage rings have the property that $\alpha_0 > 0$, implying that particles with larger energy will travel along longer closed orbits with more radial excursions. A longer closed orbit may imply relatively longer revolution period T. On the other hand, a higher energy particle travels with higher velocity v and the period of revolution will be relatively shorter. The result is a slip in revolution time ΔT (either positive or negative) every turn with respect to the on-momentum particle. The particles inside the bunch will therefore spread out longitudinally and the bunch will disintegrate unless there is some longitudinal focusing force like the rf voltage. Since $T = C/v$, a *slip factor* η can be defined by

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C_0} - \frac{\Delta v}{v_0} \equiv \eta\delta \ , \tag{2.3}$$

where T_0 is the revolution period of the on-momentum particle. Thus, to the lowest order in the fractional momentum spread, we have

$$\eta = \alpha_0 - \frac{1}{\gamma_0^2} \ , \tag{2.4}$$

where $E_0 = \gamma_0 mc^2$ and m is the rest mass of the particle. Higher orders of the slip factor will be given in Chapter 18.

For most electron rings and high energy proton rings, the particle velocity v is extremely close to c, the velocity of light, so that the revolution-time slip is dominated by the increase in orbit length. We therefore have $\eta \approx \alpha_0$ and we call the operation *above the transition energy*. For low-energy hadron rings, the velocity term may dominate making $\eta < 0$ and we say the operation is *below the transition energy*, implying that the velocity change of an off-momentum particle is more important than the change in orbit length. The higher-momentum particle, having a larger velocity, will complete a revolution turn in less time than the on-momentum particle, resulting in a forward slip. Obviously, transition occurs when the velocity change is just as important as the change in orbit length, or $\eta = 0$. The transition energy is defined as $E_t = \gamma_t mc^2$ and $\gamma_t = \alpha_0^{-1/2}$. There are also rings, like the 1.2 GeV CERN Low Energy Antiproton Ring (LEAR) and many newly designed ones [2] that have negative momentum-compaction factors or $\alpha_0 < 0$. In these rings, lower momentum particles have longer closed orbits or larger radial excursions than higher momentum particles. Negative momentum compaction implies an imaginary γ_t and the slip factor will always be negative, indicating that the ring will be always below transition. Some believe that such rings will be more stable against collective instabilities [3]. Design and study of negative momentum compaction rings have been an active branch of research in accelerator physics lately [4].

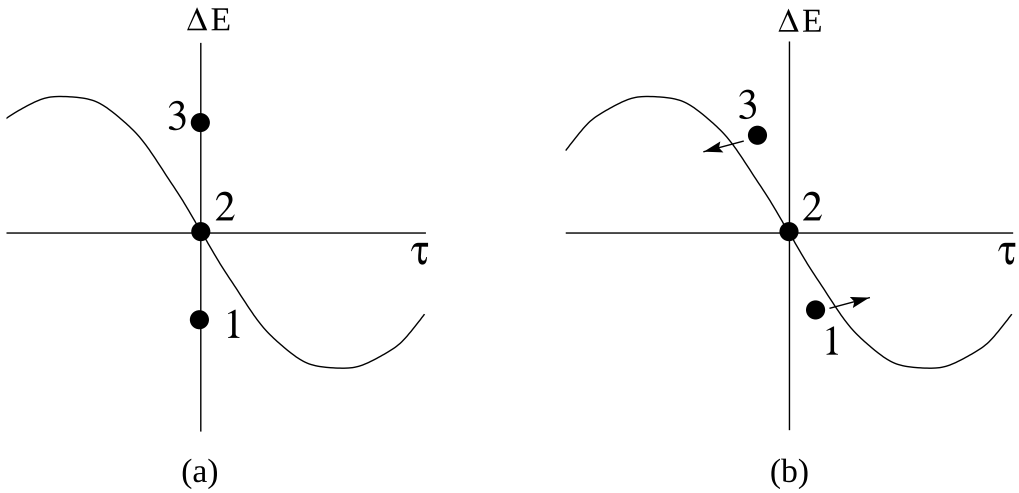

Figure 2.1: Three particles are shown in the longitudinal phase planes. (a) Initially, they are all at the rf phase of 180° and do not gain or lose any energy. (b) One turn later, the on-momentum particle, denoted by 2, arrives with the same phase of 180° without any change in energy. The particle with lower energy, denoted by 1, arrives earlier and gains energy from the positive part of the rf voltage wave at phase $< 180°$. The particle with higher energy, denoted by 3, arrives late and loses energy because it sees the rf voltage wave at phase $> 180°$.

In order to have the particles bunched, a longitudinal focusing force will be required. This is done by the introduction of rf cavities. Consider three particles arriving in the first turn at exactly the same time at a cavity gap, where the rf sinusoidal gap voltage wave is at 180°, as shown in Fig. 2.1a. All three particles are seeing zero rf voltage and are not gaining any energy from the rf wave. The drawing of the rf voltage wave implies that the rf voltage at the cavity gap was positive a short time ago and will be negative a short time later. Assume that the ring is above transition or $\eta > 0$. One turn later, the on-momentum particle, denoted by 2 in the figure, arrives at the cavity gap at exactly the time when the rf sinusoidal voltage curve is again at 180° and gains no energy. At this moment, the positions of the three particles and the rf wave are shown in Fig. 2.1b. The lower energy particle, denoted by 1, arrives at the gap earlier by τ_1, which we call *time slip*. It sees the positive part of the rf voltage and gains energy. For the second turn, it arrives at the gap earlier by $\tau_1 + \tau_2$, where $\tau_2 < \tau_1$ because the particle energy has been raised in the second passage. This particle will continue to gain energy from the rf every turn and its turn-by-turn additional time slip diminishes. Eventually, this particle will have an energy higher than the on-momentum particle and starts to arrive at the cavity gap later turn after turn, or its turn-by-turn time slip becomes negative. Similar conclusion can be drawn for the particle, denoted by 3 in the figure, that has initial energy higher than the on-momentum particle. With the rf voltage wave, the off-momentum particles will oscillate around the on-momentum particle and continue to form a bunch. In reality, the particles lose an amount of energy U_s every turn due to synchrotron radiation. This is compensated by shifting the rf phase slightly from 180° to $\phi_s = \sin^{-1}(U_s/V_{\rm rf})$, where $V_{\rm rf}$ is the rf voltage (the peak value of the rf wave), so that the on-momentum particle will see the rf voltage at the phase ϕ_s when traversing the cavity gap. This particle is also called the *synchronous* particle.

2.2 Equations of Motion

To measure the charge distribution in a bunch, we choose a fixed reference point s_0 along the ring and put a detector there. A particle in a bunch is characterized longitudinally by τ, the time it arrives at s_0 ahead of the synchronous particle. We record the amount of charge arriving when the time advance is between τ and $\tau + d\tau$. The result is $e\rho(\tau)d\tau$, where $\rho(\tau)$ is a measure of the particle distribution* and e is the particle charge. The actual linear particle density per unit length is $\lambda(\tau) = \rho(\tau)/v$, where v is the velocity of the synchronous particle. Note that this charge distribution is measured at a fixed point but at different times. Therefore, it is *not* a periodic function of τ. In one turn, the change in time advance is

$$\Delta\tau = -\eta T_0 \delta \ . \tag{2.5}$$

The negative sign comes about because the period of a higher-momentum particle is larger above transition ($\eta > 0$) and therefore its time of arrival slips. During that turn, the energy gained by the particle relative to the synchronous particle is

$$\Delta E = eV_{\rm rf}(\sin\phi - \sin\phi_s) - [U(\delta) - U_s] + C\langle F_0^{\|}\rangle - C_0\langle F_{0s}^{\|}\rangle \ , \tag{2.6}$$

where the subscript s stands for synchronous particle. The first term on the right is the sinusoidal rf voltage and the second term is the radiation energy. The third is the average wake force defined in the previous section due to all beam particles ahead; it can therefore be written as, according to Fig. 2.2,

*Note the change in notation. In Chapter 1, ρ represents charge density. From here on, ρ represents particle number density so that $\int \rho(\tau)d\tau = N_b$ the total number of particles in the bunch. The charge density becomes $e\rho$.

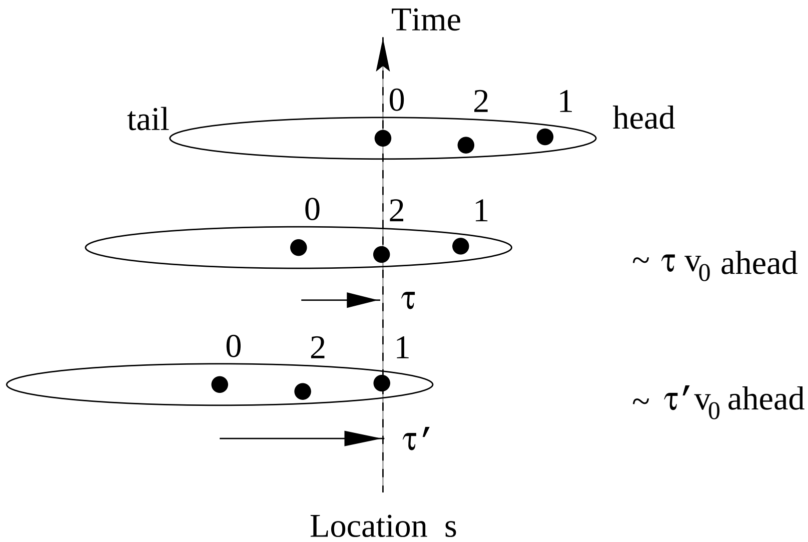

Figure 2.2: Top: the synchronous particle 0 arriving at the location s at the ring. Middle: test particle 2 arriving at s with a time advance τ and seeing the wake left by source particle 1 (bottom) arriving at s with a time advance τ'. Thus test particle is $z \sim v_0(\tau' - \tau)$ behind source particle. The total wake force acting on test particle 2 is the superposition of the wake forces contributed by all particles in the bunch with time advances $\tau' \geq \tau$.

$$\langle F_0^{\|}(\tau)\rangle = -\frac{e^2}{C}\int_\tau^\infty d\tau' \rho(\tau') W_0'(\tau' - \tau) \ . \tag{2.7}$$

Notice that we have written, for convenience, the wake function as a function of time advance $(\tau' - \tau)$ instead of distance $z \sim v_0(\tau' - \tau)$, with v_0 denoting the velocity of the synchronous particle. There is an approximation here because the particles inside the bunch travel with slightly different velocity. The error, which is less then $\tau_L \Delta v$, is small, where τ_L is the total bunch length in time and Δv is the maximum velocity spread in the bunch. This is actually the rigid-bunch approximation. In the same approximation, we do not distinguish between C and C_0, the path length of an off-momentum particle and that of the synchronous particle. The signs in Eq. (2.7) and in front of $\langle F_0^{\|}(\tau)\rangle$ in Eq. (2.6) can be checked by seeing whether there is an energy loss when substituting, the wake of, for example, a real resistance $W_0'(\tau) = R\delta(\tau)$. The synchronous phase ϕ_s in Eq. (2.6) is a parameter chosen to balance the energy loss in a storage ring, or to accomplish a designed rate of increase of energy in an accelerator. The average wake force acting on the synchronous particle, $\langle F_{0s}^{\|}\rangle$, can be obtained from Eq. (2.7) by letting $\tau = 0$.

The two equations of motion are related because the momentum spread is related to the energy spread by $\delta = \Delta E/(\beta_0^2 E_0)$, and the rf phase seen is related to the time advance,

$$\phi - \phi_s = -h\omega_0\tau \ , \tag{2.8}$$

where $\omega_0/(2\pi) = 1/T_0$ is the revolution frequency of the ring for the synchronous particle and h is the rf harmonic, which is the number of oscillations the rf wave makes during one revolution period. The negative sign on the right side of Eq. (2.8) comes about because when the particle arrives earlier or $\tau > 0$, it sees a rf phase *earlier* than the synchronous phase ϕ_s (see Fig. 2.1). Writing as *discrete* differential equations, they become

$$\frac{d\tau}{dn} = -\frac{\eta T_0}{\beta_0^2}\frac{\Delta E}{E_0} \ , \tag{2.9}$$

$$\frac{d\Delta E}{dn} = eV_{\rm rf}[\sin(\phi_s - h\omega_0\tau) - \sin\phi_s] - [U(\delta) - U_s] + C_0\left(\langle F_0^{\|}\rangle - \langle F_{0s}^{\|}\rangle\right) \ . \tag{2.10}$$

To simplify future mathematical derivations, a continuous independent variable is needed instead of the discrete turn number n. Time is not a good variable here because particles with different energies complete one revolution turn in different time intervals. Even for one particle, its energy oscillates with synchrotron motion and so is the time for consecutive revolution turns. We choose instead s, the distance measured along the closed orbit of the synchronous particle, because the increase in s per revolution turn is always the length of the closed orbit† C_0 of the synchronous particle, regardless of the momentum offset of the beam particle under consideration. This transition from discrete turn number n to the continuum is a good approximation, because in reality it takes a particle many ($\sim$ 50 to 100 in electron rings and $\sim$ 200 to 1000 in proton rings) revolution turns to complete a synchrotron oscillation, and it takes the beam a large number of turns for an instability to develop.

With τ and ΔE as the canonical variables‡, the equations of motion for a particle in a small bunch become

$$\frac{d\tau}{ds} = -\frac{\eta}{v_0\beta_0^2 E_0}\Delta E \ , \tag{2.11}$$

$$\frac{d\Delta E}{ds} = \frac{eV_{\rm rf}}{C_0}\Big[\sin(\phi_s - h\omega_0\tau) - \sin\phi_s\Big] - \frac{U(\Delta E) - U_s}{C_0} + \langle F_0^{\|}(\tau, s)\rangle - \langle F_{0s}^{\|}(s)\rangle \ . \tag{2.12}$$

†In subsequent chapters, the subscript '0' in C_0, E_0, v_0, β_0, γ_0, etc for the synchronous particle may be omitted in order to simplify the notation.

‡This set of canonical variables should not be used if the accelerator is ramping. Instead the set τ/ω_0 and $\Delta E/\omega_0$ is preferred

Although one may also use $t = s/v_0$ as the independent variable, we want to emphasize that this t is the time describing the synchronous particle and is *not* the time variable for an off-energy particle. Thus, the independent variable s is quite different from the time variable.

Let us first neglect the wake potential and also the small difference between the energy lost by the off-momentum particle $U(\delta)$ and the energy lost by the on-momentum particle U_s. For small amplitude oscillations, the two equations combine to give

$$\frac{d^2\tau}{ds^2} - \frac{2\pi\eta heV_{\rm rf}\cos\phi_s}{C_0^2\beta_0^2E_0}\tau = 0\ . \tag{2.13}$$

Therefore, the bunch particles are oscillating with the angular frequency $\omega_{s0} = \nu_{s0}\omega_0$, where

$$\nu_{s0} = \sqrt{-\frac{\eta heV_{\rm rf}\cos\phi_s}{2\pi\beta_0^2E_0}} \tag{2.14}$$

is called the *synchrotron tune* or the number of synchrotron oscillations a particle makes in one revolution turn, and $\omega_{s0} = \nu_{s0}\omega_0$ the *synchrotron angular frequency*. The subscript "0" indicates that these are the *unperturbed* small-amplitude values or with the wake potential turned off. The negative sign inside the square root implies that ϕ_s should be near 180° in the second quadrant above transition ($\eta > 0$), but near 0° in the first quadrant below transition ($\eta < 0$). Synchrotron motion is slow and the synchrotron tune is usually of the order of 0.001 to 0.01. When the oscillation amplitude becomes larger, the rf sine wave cannot be linearized. The focusing force is smaller and the synchrotron tune ν_s for maximum phase excursion $\hat\phi$ will become smaller as is shown in Fig. 2.3 according to

$$\nu_s(\hat\phi) = \frac{\pi\nu_{s0}}{2K(\sin\frac{1}{2}\hat\phi)}\ , \tag{2.15}$$

where

$$K(x) = \int_0^{\pi/2}\frac{du}{\sqrt{1-x^2\sin^2u}} \tag{2.16}$$

is the complete elliptic integral of the first kind. This prediction has been verified experimentally at the Indiana University Cyclotron Facility (IUCF) Cooler Ring [5]. In the small-amplitude approximation, we have $\nu_s(\hat\phi) = \nu_{s0}\left(1 - \frac{1}{12}\hat\phi^2\right)$. In other words, there will be a spread in the synchrotron tune among the particles in the bunch, which will be very essential to the Landau damping of the collective instabilities to be discussed later. As the oscillation amplitude continues to increase, a point will be reached when

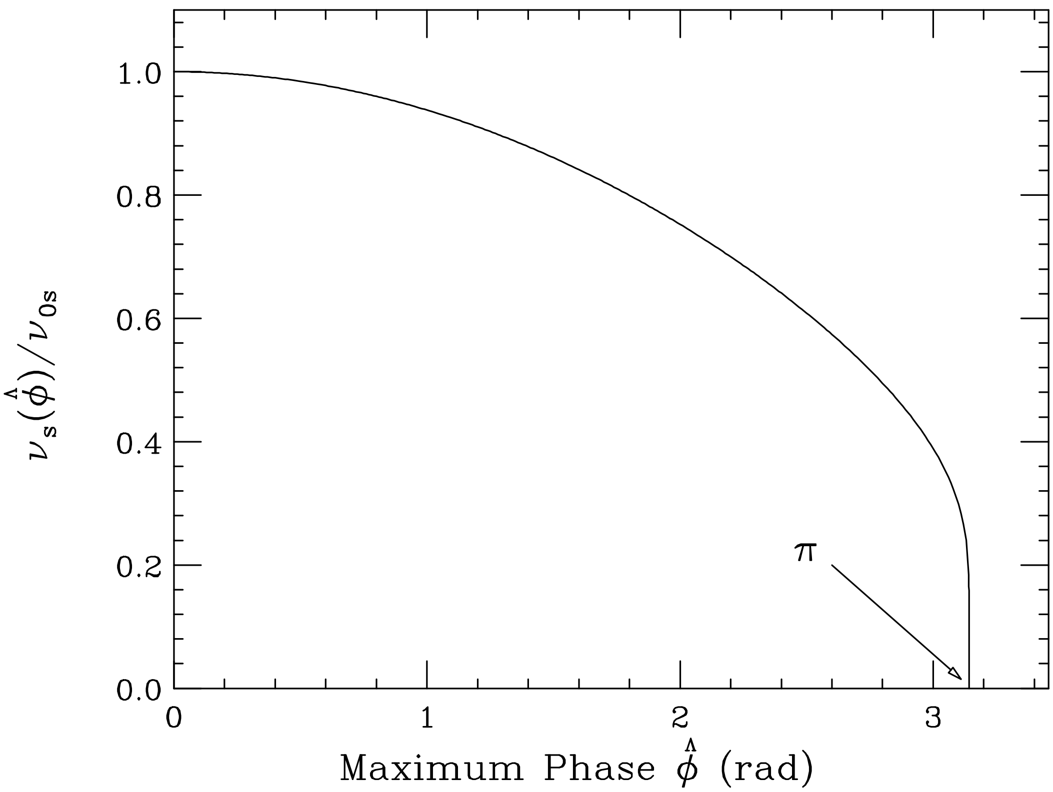

Figure 2.3: Plot showing the synchrotron frequency decreasing to zero at the edge of the rf bucket.

there is no more focusing provided by the rf voltage anymore. This boundary in the τ-ΔE phase space provides the maximum possible bunch area allowed and is called the *rf bucket* holding the bunch. Any particle that goes outside the bucket will be lost. The equation of motion is, in fact, exactly that of a pendulum, whose frequency of oscillation decreases with amplitude. If we start the pendulum motion at its rest position with too large a kinetic energy, the pendulum will no longer be in oscillatory motion. It will wrap around the point of support performing librations instead. This critical angular amplitude of the pendulum is $\pm\pi$, exactly the same for the rf bucket. Figure 2.4 illustrates some stationary buckets (when the synchronous phase $\phi_s = 180°$ above transition) and moving or accelerating buckets (when ϕ_s is between 90° and 180°). The figure also shows the trajectories of libration outside the buckets. The horizontal axis is the rf phase (instead of the time advance used in Fig. 2.1); the trajectories therefore move clockwise (instead of counter-clockwise in Fig. 2.1).

If the radiation energy is neglected, the two equations of motion are derivable from the Hamiltonian

$$H = -\frac{\eta}{2v_0\beta_0^2 E_0}(\Delta E)^2 - \frac{eV_{\rm rf}}{C_0 h\omega_0}\left[\cos(\phi_s - h\omega_0\tau) - \cos\phi_s - h\omega_0\tau\sin\phi_s\right] + V(\tau)\ , \tag{2.17}$$

with the aid of the Hamiltonian equations

$$\left\{\begin{aligned} \frac{d\tau}{ds} &= \frac{\partial H}{\partial \Delta E}\ , \\ \frac{d\Delta E}{ds} &= -\frac{\partial H}{\partial \tau}\ . \end{aligned}\right. \tag{2.18}$$

The potential of the wake force is given by

$$V(\tau) = \frac{e^2}{C_0}\int_0^\tau d\tau'' \left[\int_{-\infty}^{\infty} d\tau' \rho(\tau') W_0'(\tau' - \tau'') - \int_{-\infty}^{\infty} d\tau' \rho(\tau') W_0'(\tau')\right]\ . \tag{2.19}$$

The second term in the squared brackets comes from $\langle F_{0s}^{\parallel}\rangle$, the energy lost by the synchronous particle due to the wake potential of the vacuum chamber. In Eq. (2.17), the $\cos\phi_s$ term is added to adjust the rf potential to zero for synchronous particles ($\tau = 0$). For small-amplitude oscillations, the Hamiltonian simplifies to

$$H = -\frac{\eta}{2v_0\beta_0^2 E_0}(\Delta E)^2 - \frac{\omega_{s0}^2\beta_0^2 E_0}{2\eta v_0}\tau^2 + V(\tau)\ , \tag{2.20}$$

where $\omega_{s0} = \nu_{s0}\omega_0$, the synchrotron angular frequency for small amplitudes, is given by Eq. (2.14).

In an electron ring, synchrotron radiation may provide damping to many collective instabilities. Because this damping force is dissipative in nature, strictly speaking a Hamiltonian formalism does not apply. However, the synchrotron radiation damping time is usually very much longer than the synchrotron period. The fast growing instabilities will evolve to their full extent before the damping mechanism becomes materialized. Here, we are interested mostly in studying those instabilities that grow within one radiation damping time of the ring. For a time period much less than the radiation damping time, radiation can be neglected and the Hamiltonian formalism therefore applies.

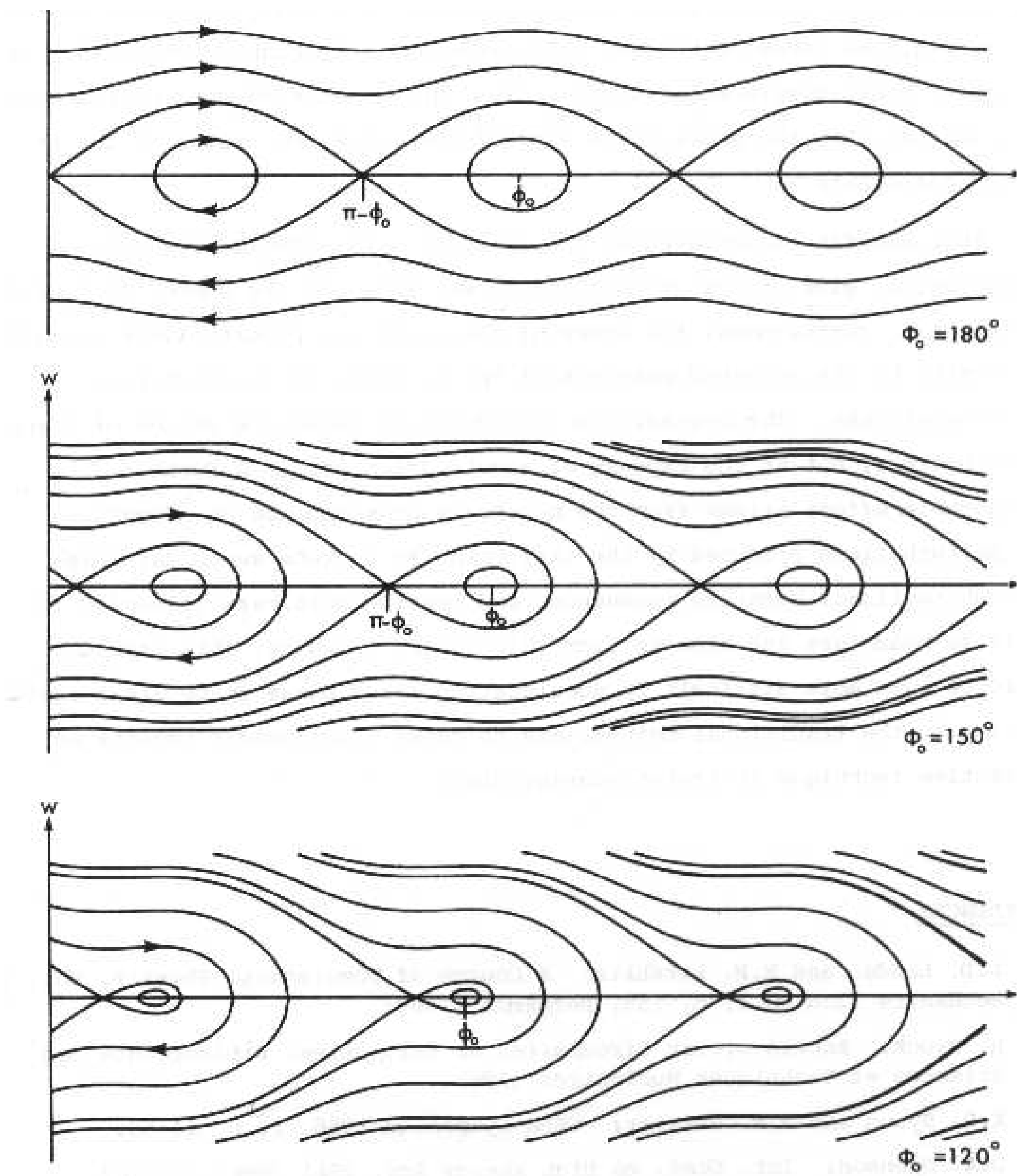

Figure 2.4: The trajectories in the longitudinal phase space above the transition energy. Top plot shows the stationary buckets when the synchronous phase $\phi_0 = 180°$. Middle and lower plots show the moving or accelerating buckets when the synchronous phases are, respectively, $\phi_0 = 150°$ and $120°$. The moving buckets shrink when the synchronous phase decreases from $180°$ towards $90°$. Notice that the horizontal axis is the rf phase (instead of arrival time in Fig. 2.1); the directions of the trajectories are therefore clockwise above transition.

2.3 Vlasov Equation

We would like to study the evolution of a bunch that contains, say, 10^{12} particles. The Hamiltonian in Eq. (2.17) has to be modified to include 10^{12} sets of canonical variables in order to fully describe the bunch. The description of the motion of a collection of 10^{12} particles is known as the *particle approach*, and is often tackled in the time domain. However, what are of interest to us are the collective behaviors of the bunch like the motion of its centroid, the evolution of the particle distribution, etc. In other words, we are studying here the evolution of various modes of motion of these collective variables. For 10^{12} particles, there are 10^{12} modes of motion. However, we will never be interested in those modes whose wavelengths are of the order of the separation between two adjacent particles inside the bunch, because they correspond to motions of very high frequencies, and those motions are microscopic in nature. What we would like to study are the macroscopic modes of the bunch, or those having wavelengths of the same order as the length of the bunch or the radius of the vacuum chamber. Sometimes, we may even want to study modes with wavelengths one tenth or one hundredth of the bunch length or beam pipe radius, but definitely not down to the microscopic size like the distance between two neighboring beam particles. In other words, we go to the frequency domain and look at the different modes of motion of oscillation of the bunch as a whole. Our interest is on those few modes that have the lowest frequencies or longest wavelengths. This direction of study is known as the *mode approach*.

When collisions are neglected, the basic mathematical tool for the mode approach is the Vlasov equation or the Liouville theorem [6]. It states that if we follow the motion of a representative particle in the longitudinal or τ-ΔE phase space, the density of particles in its neighborhood is constant. In other words, the distribution of particles $\psi(\tau,\Delta E;s)$ moves in the longitudinal phase space like an incompressible fluid. Mathematically, the Vlasov equation reads

$$\frac{d\psi}{ds}=\frac{\partial\psi}{\partial s}+\frac{d\tau}{ds}\frac{\partial\psi}{\partial\tau}+\frac{d\Delta E}{ds}\frac{\partial\psi}{\partial\Delta E}=0\ . \tag{2.21}$$

In terms of the Hamiltonian, it becomes

$$\frac{\partial\psi}{\partial s}+[\psi,H]=0\ , \tag{2.22}$$

where $[,]$ denotes the Poisson bracket. Here, the time of early arrival τ and the energy offset ΔE are the set of canonical variables chosen. The Poisson bracket is therefore

$$[\psi,H]=\frac{\partial\psi}{\partial\tau}\frac{\partial H}{\partial\Delta E}-\frac{\partial\psi}{\partial\Delta E}\frac{\partial H}{\partial\tau}\ . \tag{2.23}$$

Together with the Hamiltonian equations of Eq. (2.18), Eq. (2.21) is reproduced.

If radiation is included in the discussion, one must extend the Vlasov equation to the Fokker-Planck equation [7]

$$\frac{d\psi}{ds} = A\frac{\partial}{\partial\Delta E}(\Delta E\psi) + \frac{D}{2}\frac{\partial^2\psi}{\partial\Delta E^2} \ , \tag{2.24}$$

where A and D are related, respectively, to the damping and diffusion coefficients.

2.4 Coasting Beams

A coasting beam is not bunched. There is no rf voltage and therefore no synchrotron oscillation. Thus, there is no synchronous particle. For the longitudinal position, we can make reference with respect to a designated point in the accelerator ring. For the energy offset, we can make reference with respect to the average energy change for all the on-momentum particles. Here, we cannot talk about bunch modes. Instead, the linear density of an excitation of the beam can be described much better by an harmonic wave,

$$f_1(\theta,t) \sim e^{in\theta-\Omega t} \ , \tag{2.25}$$

where θ is the azimuthal angle around the ring measured from a point of reference, n is a revolution harmonic or n modulations of the longitudinal linear density when viewed from the top of the accelerator ring at a fixed time t, and Ω is the angular velocity of the wave. The harmonic $n=0$ should be excluded because it will violate charge conservation since the integral of f_1 over the whole ring does not vanish when $n=0$. The excitation of Eq. (2.25) is a *snap-shot* view, similar to taking a picture of the beam above the accelerator ring. Thus the linear density is a periodic function of θ with period 2π. The linear density can therefore be expanded as a Fourier series and the excitation $f_1(s,t)$ is just a Fourier component. To describe a beam particle, we use the canonical variable z and ΔE, where $z=R\theta$ with $R=C_0/(2\pi)$ being the mean radius of the on-momentum closed orbit. Here, z is just the longitudinal distance ahead of the point of reference at time t and ΔE is the energy offset. Since we are using snap-shot description, the real time t can be used as the continuous independent variable. The equations of motion are

$$\frac{dz}{dt} = -\frac{\eta\Delta E}{v_0\beta_0^2E_0} \ , \tag{2.26}$$

$$\frac{d\Delta E}{dt} = -\frac{U-U_s}{T_0} + v_0\langle F_0^{\parallel}(z,t)\rangle - v_0\langle F_{0s}^{\parallel}(t)\rangle \ , \tag{2.27}$$

where v_0 and T_0 are, respectively, the velocity and revolution period of the on-momentum particle, $\langle F_0^{\|}(z,t)\rangle$ is the average longitudinal wake force acting on the beam particle under consideration and $\langle F_{0s}^{\|}(t)\rangle$ is the average of the average longitudinal wake force acting on all the on-momentum particles. The subtraction of $\langle F_{0s}^{\|}(t)\rangle$ is necessary, because sometimes the average wake force may have a dc resistive term and we do not want to include it in our discussion since it is usually compensated, for example, by a dc gap voltage. Otherwise, the beam will continue to lose energy and will not be able to stay inside the vacuum chamber.

When synchrotron radiation is neglected, the equations of motion can be derived from the Hamiltonian

$$H = -\frac{\eta\Delta E}{2v_0\beta_0^2 E_0} + \int_0^z \left[\langle F_0^{\|}(z',t)\rangle - \langle F_{0s}^{\|}(t)\rangle\right] v_0 dz' \ . \tag{2.28}$$

For the beam distribution $\psi(z,\Delta E;t)$ in the longitudinal phase space, the Vlasov equation becomes

$$\frac{\partial\psi}{\partial t} + \frac{dz}{dt}\frac{\partial\psi}{\partial z} + \frac{d\Delta E}{dt}\frac{\partial\psi}{\partial\Delta E} = 0 \ , \tag{2.29}$$

where dz/dt and $d\Delta E/dt$ are given by the equations of motion. It is important to realize that dz/dt is *not* the longitudinal velocity v of the particle having energy offset ΔE. Instead, it represents the phase slip (in length) per revolution period T_0.

2.5 Exercises

2.1. The Hamiltonian of Eq. (2.17) describes motion in the longitudinal phase space, when the wake potential $V(\tau)$ is not included. With the effects of the wake potential neglected, find the fixed points of the Hamiltonian above and below transition, and determine whether they are stable or not. The separatrices are the contours of fixed Hamiltonian values that pass through the unstable fixed points. They separate the region of libration motion from rotation motion[§]. Plot the separatrices.

2.2. The canonical variables τ_0 and ΔE_0 evaluated at 'time' $s=0$ become τ_1 and ΔE_1 at an infinitesimal time Δs later according to

$$\tau_1 = \tau_0 + \frac{\partial H}{\partial \Delta E_0}\Delta s\,, \qquad \Delta E_1 = \Delta E_0 - \frac{\partial H}{\partial \tau_0}\Delta s\,. \tag{2.30}$$

Consider the small phase-space area element $d\tau_0 d\Delta E_0 = J d\tau_1 d\Delta E_1$. Show that the Jacobian $J=1$ to the first order in Δs, implying that the area surrounding a given number of particles does not change in time, which is Liouville Theorem. It is possible to prove $J=1$ to all orders in Δs using canonical transformation. See, for example, H. Goldstein, *Classical Mechanics*, Addison-Wesley, Chapter 8-3.

2.3. Starting from the Hamiltonian in Eq. (2.17) with the synchronous phase $\phi_s = 0$ or π but in the absence of the wake potential, derive the synchrotron tune, Eq. (2.3), of a particle having an rf phase amplitude $\hat{\phi}$. Repeat the derivation for any arbitrary synchronous phase.

[§]Libration implies periodic motion in the phase space, similar to a sine wave going from $-\infty$ to $+\infty$. Rotation motion in phase space implies to-and-fro oscillatory motion.

Bibliography

[1] C. Bernardine, *et al*, Phys. Rev. Lett. **10**, 407 (1963).

[2] *JHP Accelerator Design Study Report*, KEK Report 97-16, JHF-97-10, 1997; K.Y. Ng, D. Trbojevic, and S.Y. Lee, Proceedings of 1991 IEEE Particle Accelerator Conference, San Francisco, 1991, Ed. Lizama (IEEE, New York, 1991), pp. 159; C. Ankenbrandt, *et al*, Phys. Rev. ST Accel. Beams 1, 081001 (1999).

[3] S.X. Fang, K. Oide, K. Yokoya, B. Chen, J.Q. Wang, Proceedings of 1995 Particle Accelerator Conference, Dallas, pp. 3064.

[4] S.Y. Lee, K.Y. Ng, and D. Trbojevic, Phys. Rev. **E4**, 3040 (1993);S.Y. Lee, K.Y. Ng, and D. Trbojevic, Fermilab Internal Report FN-595, 1992.

[5] H. Huang, et al., Phys. Rev. **E48**, 4678 (1993).

[6] A.A. Vlasov, J. Phys. USSR **9**, 25 (1945).

[7] H. Risken, *The Fokker-Planck Equation*, 2nd Edition, Springer-Verlag, N.Y., 1989.